

ON THE POYNTING-ROBERTSON EFFECT AND ANALYTICAL SOLUTIONS

Jozef Klačka

Institute of Astronomy, Faculty for Mathematics and Physics, Comenius University,
Mlynská dolina, 842 28 Bratislava, Slovak Republic

Abstract. Solutions of the two-body problem with the simultaneous action of the solar electromagnetic radiation in the form of the Poynting-Robertson effect are discussed. Special attention is devoted to pseudo-circular orbits and terminal values of osculating elements. The obtained results complete those of Klačka and Kaufmannová (1992) and Breiter and Jackson (1998).

Terminal values of osculating elements presented in Breiter and Jackson (1998) are of no physical sense due to the fact that relativistic equation of motion containing only first order of v/c was used in the paper.

Key words: celestial mechanics, stellar dynamics

1. Introduction

Breiter and Jackson (1998; BJ-paper in the following text) have presented analytical mathematical solutions of two-body problems with drag. The Poynting-Robertson effect (P-R effect) is one of the special cases discussed in the BJ-paper.

The conclusion of the BJ-paper concerns the fact that analytical solution in terms of special functions exists for the P-R effect. However, this result is useful only in the two limiting cases: pseudo-circular orbits and terminal values of osculating elements. As for the general case, there is a complication with calculation of infinite functional series and even with the divergence of the series.

The problem of the divergence of the infinite functional series can be overcome by numerical integration of the equation(s) of motion(s). This can be easily done at the present epoch of computers. However, it is nice, indeed, when a man can compare his numerical calculations with analytical solutions in the limiting cases – analytical solutions are very interesting.

The aim of this paper is to discuss analytical solutions for the P-R effect in the two limiting cases: pseudo-circular orbits and terminal values of osculating elements. It is shown that the real results do not completely correspond to the results presented in the BJ-paper. General analytical solutions for pseudo-circular orbits presented in our paper are presented.

2. Overview of the correctly presented results

The first correct special result for the P-R effect was presented by Robertson (1937; cited in accordance with BJ-paper). The complete formula for the P-R effect is correctly derived in Klačka (1992a). Other, more simple correct derivations may be found in Klačka's papers: 1992b, 1993a, 1993b. These papers present also arguments as for physical incorrectness in papers cited in the BJ-paper (except of Robertson).

As for the solutions of the equation of motion for the P-R effect, we refer also to Klačka (1992c; error is in Eq. (10) of the paper – the right-hand side of Eq. (10) must contain $\mu (1 - \beta)$ instead of μ when used in the section 3), Klačka (1993c, 1993d), Klačka and Kaufmannová (1992, 1993 – typewriting error in Eq. (1)) (see also Klačka 1994a). As for other papers of the author, dealing mainly with general interaction of the electromagnetic radiation (of the Sun; and, also, solar corpuscular radiation), we refer to: Klačka (1993f, 1993g (some numerical errors which may be easily found are in the last section; moreover, real particle should rotate around one axis – axis of rotation), 1994b), Klačka and Kocifaj (1994).

3. P-R effect and analytical solutions

At first, we put Eq. (18) in BJ-paper into a correct form:

$$\dot{r} = \frac{\mu}{\alpha_t} x^\nu (\nu y + x y') . \quad (1)$$

In the case of the P-R effect we have $\nu = 1$, μ must be substituted by $\mu_0 (1 - \beta)$ and $\alpha_t = \beta \mu_0 / c$:

$$\begin{aligned} \dot{r} &= c \frac{1 - \beta}{\beta} x (y + x y') , \\ \alpha_t &= \beta \mu_0 / c . \end{aligned} \quad (2)$$

Let the initial orbit is given by r_{in} , \dot{r}_{in} , h_{in} and $\vartheta_{in} = 0$. Eqs. (5) in BJ-paper and Eqs.

(2) of our paper yield

$$\begin{aligned} x_{in} &= \frac{c}{\beta \mu_0} h_{in} , \\ y_{in} &= \frac{h_{in}^2}{\mu_0 (1 - \beta) r_{in}} x_{in}^{-3} , \\ y'_{in} &= \left\{ \frac{\dot{r}_{in}}{c} \frac{\beta}{1 - \beta} x_{in}^2 - \frac{h_{in}^2}{\mu_0 (1 - \beta) r_{in}} \right\} x_{in}^{-4} . \end{aligned} \quad (3)$$

These equations complete Eqs. (16) in BJ-paper.

4. Special types of pseudo-circular orbits

4.1. Increasing eccentricity

Due to the fact that Bessel's functions $J_1(x)$, $Y_1(x)$ can be expressed as linear combinations of $\cos(x - 3\pi/4)$, $\sin(x - 3\pi/4)$, Eqs. (16) in BJ-paper enable to fulfil the conditions $A = B = 0$. These conditions are fulfilled by conditions

$$\begin{aligned} y_{in} &= \hat{S}_1(x_{in}) , \\ y'_{in} &= \hat{S}'_1(x_{in}) . \end{aligned} \quad (4)$$

Eqs. (3) and (4) lead to special type of pseudo-circular orbits, as it is discussed in section 5 in BJ-paper. The procedure goes in the way that giving h_{in} we can calculate $x_{in} = h_{in} c / (\beta \mu_0)$; the first of Eqs. (4) and second of Eqs. (3) yield

$$r_{in} = \frac{\beta^3}{1 - \beta} \frac{\mu_0^2}{h_{in}} \left\{ \hat{S}_1(x_{in}) \right\}^{-1} ; \quad (5)$$

the second of Eqs. (4) and the third of Eqs. (3) yield

$$\dot{r}_{in} = c \frac{1 - \beta}{\beta} x_{in}^{-2} \left\{ x_{in}^4 \hat{S}'_1(x_{in}) + \frac{h_{in}^2}{\mu_0 (1 - \beta) r_{in}} \right\} . \quad (6)$$

Transversal component of velocity may be calculated from the relation $v_{T\ in} = h_{in} / r_{in}$.

The case $e \approx 2/x$ (Eq. (28) in BJ-paper), together with $\sqrt{\mu_0 (1 - \beta) p} = x \beta \mu_0 / c$, leads to

$$p e^2 \approx \frac{(2\beta)^2}{1 - \beta} \frac{\mu_0}{c^2} . \quad (7)$$

In any case the conclusion about the increasing eccentricity is not completely correct. The eccentricity begin to oscilate at some state of the orbital evolution, although its mean value is an increasing function of time. The conclusion "The general conclusion is that

in the pseudo-circular solution the osculating eccentricity *grows systematically* from the small but nonzero value of order α_t/h towards $e = 1$.” (BJ-paper, section 5) is incorrect (see also section 4 of our paper).

4.2. Oscillating eccentricity

Pseudo-circular orbit discussed in BJ-paper is only a very special case of possible pseudo-circular orbits. We complete the case by all the other possibilities, where also zero value of eccentricity is possible.

Let us consider a situation when meteoroid is ejected from comet at cometary aphelion with zero ejection velocity. If the comet’s orbital elements are a_0 , e_0 and meteoroid’s β is given by the condition $\beta = e_0$, then Eqs. (30) and (31) in Klačka (1992c) and Eq. (17) in Klačka (1993e; Eq. (17) must contain $\mu_0 (1 - \beta)$ instead of μ , now) yield

$$\begin{aligned} e_{in} &= 0 , \\ a_{in} &= a_0 (1 + e_0) , \\ H_{in} &= \sqrt{\mu_0 a_0 (1 - e_0^2)} . \end{aligned} \tag{8}$$

(If meteoroid is ejected with nonzero velocity, one can use equations of Gajdošík and Klačka (1999).) Then, we have

$$\begin{aligned} r_{in} &\equiv r_0 = a_0 (1 + e_0) , \\ \dot{r}_{in} &\equiv \dot{r}_0 = 0 . \end{aligned} \tag{9}$$

If we set $\vartheta_0 = 0$, $\alpha_t = \beta \mu_0 / c$, equations of sections 2, 3 and 4 of BJ-paper yield

$$\begin{aligned} h &= \sqrt{\mu_0 a_0 (1 - e_0^2)} , \\ x_{in} &\equiv x_0 = h \alpha_t^{-1} = (c / e_0) \sqrt{\mu_0 (1 - e_0^2) / a_0} , \\ y_{in} &\equiv y_0 = \{\mu_0 (1 - \beta) r_{in} x_0\}^{-1} \alpha_t^2 = x_0^{-3} . \end{aligned} \tag{10}$$

Eq. (2) yields for $\dot{r}_{in} = 0$: $y' = -y / x$, and, thus

$$y'_{in} \equiv y'_0 = -y_0 / x_0 = -x_0^{-4} . \tag{11}$$

This case should correspond to the case of Klačka and Kaufmannová (1992, 1993). We do not treat it here, since general pseudo-circular orbit will be discussed in the following section.

In order of generalizing of Eqs. (10) and (11), we use Eqs. (5) of BJ-paper:

$$\begin{aligned} x_{in} &\equiv x_0 = \frac{c}{\beta \mu_0} \sqrt{\mu_0 (1 - \beta) a_{in} (1 - e_{in}^2)}, \\ y_{in} &\equiv y_0 = \frac{\beta^2 \mu_0}{c^2 (1 - \beta) x_0} \frac{1 + e_{in} \cos f_{in}}{a_{in} (1 - e_{in}^2)} = \frac{1 + e_{in} \cos f_{in}}{x_0^3} \approx \frac{1}{x_0^3} \end{aligned} \quad (12)$$

for pseudo-circular orbits. Eq. (3) yields

$$y'_0 \approx \left\{ \frac{\sqrt{1 - \beta}}{\beta} \frac{c}{\sqrt{\mu_0 / r_{in}}} e_{in} \sin f_{in} - 1 \right\} x_{in}^{-4} \equiv - \frac{k}{x_0^4} \quad (13)$$

for pseudo-circular orbits; $y'_0 = -1 / x_0^4$ for $\dot{r}_{in} = 0$ (Eq. (11)), $y'_0 = -3 / x_0^4 + \dots$ for $A = B = 0$ (Eq. (4)).

The second of Eqs. (12) and Eq. (13) yield

$$\begin{aligned} A &\approx - \sqrt{\pi / 2} x_0^{-7/2} (3 - k) \sin(x_0 - 3\pi / 4) + \\ &\quad + \sqrt{\pi / 2} x_0^{-7/2} \frac{55 + 3k}{8 x_0} \cos(x_0 - 3\pi / 4), \\ B &\approx + \sqrt{\pi / 2} x_0^{-7/2} (3 - k) \cos(x_0 - 3\pi / 4) + \\ &\quad + \sqrt{\pi / 2} x_0^{-7/2} \frac{55 + 3k}{8 x_0} \sin(x_0 - 3\pi / 4). \end{aligned} \quad (14)$$

If $x = h \alpha_t^{-1} - \vartheta$, then

$$\begin{aligned} Z_1(x) &= A J_1(x) + B Y_1(x) \approx x_0^{-7/2} x^{-1/2} (3 - k) \sin(x - x_0) + \\ &\quad x_0^{-7/2} x^{-1/2} \left\{ \frac{9 - 3k}{8 x} + \frac{55 + 3k}{8 x_0} \right\} \cos(x - x_0), \\ \hat{S}_1(x) &\approx x^{-3} - 8 x^{-5}, \\ Z_0(x) &= A J_0(x) + B Y_0(x) \approx x_0^{-7/2} x^{-1/2} (3 - k) \cos(x - x_0) + \\ &\quad x_0^{-7/2} x^{-1/2} \left\{ \frac{3 - k}{8 x} - \frac{55 + 3k}{8 x_0} \right\} \sin(x - x_0), \\ \hat{S}'_1(x) &\approx -3 x^{-4} + 40 x^{-6}. \end{aligned} \quad (15)$$

One obtains, then

$$\begin{aligned} e \cos f &= x^3 (Z_1(x) + \hat{S}_1(x)) - 1 \approx \\ &\approx (3 - k) x_0^{-7/2} x^{5/2} \sin(x - x_0) - 8 x^{-2} + \cos(x - x_0) \times \\ &\quad \left\{ \frac{9 - 3k}{8} x_0^{-7/2} x^{3/2} + \frac{55 + 3k}{8} x_0^{-9/2} x^{5/2} \right\}, \\ e \sin f &= x^3 (Z_0(x) + x^{-1} \hat{S}_1(x) + \hat{S}'_1(x)) \approx \\ &\approx (3 - k) x_0^{-7/2} x^{5/2} \cos(x - x_0) - 2 x^{-1} + \sin(x - x_0) \times \\ &\quad \left\{ \frac{3 - k}{8} x_0^{-7/2} x^{3/2} - \frac{55 + 3k}{8} x_0^{-9/2} x^{5/2} \right\}, \end{aligned}$$

$$\begin{aligned}
& e \approx 4x_0^{-1} + (3-k)x_0^{-1}x - 4(3-k)x_0^{-1}x^2 \cos(x-x_0) \\
& + x_0^{-7/2} x^{1/2} \sin(x-x_0) \times \\
& \left\{ -\frac{33(3-k)}{2} - \frac{55+3k}{2} x_0^{-1}x + (3-k) \cos(x-x_0) \times \right. \\
& \left. \left[\frac{15-5k}{4} (x_0^{-1}x)^{7/2} - \frac{55+3k}{4} (x_0^{-1}x)^{9/2} \right] \right\}. \quad (16)
\end{aligned}$$

The equation for e^2 represents general type of evolution of eccentricity for pseudo-circular orbit in comparison with Eq. (28) in BJ-paper and papers by Klačka and Kaufmannová (1992, 1993).

5.1. Time evolution of eccentricity and true anomaly

Eqs. (16) enable us to investigate time evolution of osculating eccentricity and true anomaly. We must bear in mind that the term ‘increasing time’ corresponds to the term ‘decreasing x ’.

5.1.1. Extremes of eccentricity

The local extremes of eccentricity as a function of time are given by condition (very large values of x_0 are considered, theoretically $x_0 \rightarrow \infty$; $k \neq 3$)

$$\frac{de^2}{dx} = 0 \iff \sin(x-x_0) \approx 0. \quad (17)$$

As for the second derivative, we have

$$\frac{d^2e^2}{dx^2} \approx 4(3-k)x_0^{-7/2}x^{3/2}\cos(x-x_0). \quad (18)$$

The case $k = 3$ yields no local extreme for the leading terms. Eqs. (16) yield for evolution of true anomaly f : $3\pi/2 \rightarrow \pi$ and the value π never occurs.

The case $k = 1$ cannot be treated analytically, as it will be shown later on.

5.1.2. $x = x_0$

Putting $x = x_0$ into Eqs. (16), (17) and (18), one obtains that the point $x = x_0$ corresponds to: i) local minimum for the case $k < 3$, ii) local maximum for the case $k > 3$, for the function $e(t)$, or, $e(x)$.

If we want to find time evolution of true anomaly, we have to use Eqs. (16). The time $t = t_0 + \Delta t$, for small positive Δt , corresponds to $x = x_0 - \Delta x$, where Δx is a small positive quantity (in radians). Eqs. (16) yield, then

$$e \cos f \approx -(3-k) \frac{\Delta x}{x_0}, \quad e \sin f \approx \frac{1-k}{x_0}. \quad (19)$$

Eqs. (19) imply, on the basis of $e > 0$ and $|e \sin f| \gg |e \cos f|$.

$$\begin{aligned} \lim_{x \rightarrow x_0^-} f &= \pi / 2 \quad , \quad k < 1 \quad , \\ \lim_{x \rightarrow x_0^-} f &= 3 \pi / 2 \quad , \quad k > 1 \quad . \end{aligned} \quad (20)$$

As for the value of eccentricity, the last of Eqs. (16) yields

$$e^2(x = x_0) \approx (1 - k)^2 x_0^{-2} . \quad (21)$$

5.1.3. $x = x_0 - \pi$

Putting $x = x_0 - \pi$ into Eqs. (16), (17) and (18), one obtains that the point $x = x_0 - \pi$ corresponds to: i) local maximum for the case $k < 3$, ii) local minimum for the case $k > 3$, for the function $e(t)$, or, $e(x)$. The value of the osculating eccentricity is

$$e^2(x = x_0 - \pi) \approx (5 - k)^2 x_0^{-2} + \pi (5k - 11) (5 - k) x_0^{-3} . \quad (22)$$

If we want to find time evolution of true anomaly, we have to use Eqs. (16).

The situation shortly before the extreme corresponds to $x = x_0 - \pi + \Delta x$, where Δx is a small positive quantity (in radians) shortly before the extreme, and, Δx is a small negative quantity (in radians) shortly after the extreme. Eqs. (16) yield, then

$$e \cos f \approx - (3 - k) \frac{\Delta x}{x_0} \quad , \quad e \sin f \approx - \frac{5 - k}{x_0} . \quad (23)$$

Eqs. (23) imply

$$\begin{aligned} \lim_{x \rightarrow x_0 - \pi} f &= 3 \pi / 2 \quad , \quad k < 5 \quad , \\ \lim_{x \rightarrow x_0 - \pi} f &= \pi / 2 \quad , \quad k > 5 \quad . \end{aligned} \quad (24)$$

Eqs.(24) yield that true anomaly is a continuous function at $x = x_0 - \pi$ – at the first local maximum of osculating eccentricity.

5.1.4. $x = x_0 - 2 \pi$

Putting $x = x_0 - 2 \pi$ into Eqs. (16), (17) and (18), one obtains that the point $x = x_0 - 2 \pi$ corresponds to: i) local minimum for the case $k < 3$, ii) local maximum for the case $k > 3$, for the function $e(t)$, or, $e(x)$. As for the value of eccentricity, the last of Eqs. (16) yields

$$e^2(x = x_0 - 2 \pi) \approx (1 - k)^2 x_0^{-2} + 2 \pi (5k - 11) (5 - k) x_0^{-3} . \quad (25)$$

Comparison of Eqs. (21) and (29) leads to the conclusion

$$\begin{aligned}
e(x_0) &= e(x_0 - 2\pi), \quad k = 11/5, 5 \\
e(x_0) &< e(x_0 - 2\pi), \quad 11/5 < k < 5 \\
e(x_0) &> e(x_0 - 2\pi), \quad k < 1, 1 < k < 11/5, 5 < k
\end{aligned} \tag{26}$$

Eq. (25) shows that the case $k = 1$ leads to inconsistencies: $e(x = x_0 - 2\pi) = 0$ – only the leading term can be considered.

If we want to find time evolution of true anomaly, we have to use Eqs. (16).

The situation near extreme corresponds to $x = x_0 - 2\pi + \Delta x$, where Δx is a small positive quantity (in radians) shortly before the extreme, and, Δx is a small negative quantity (in radians) shortly after the extreme. Eqs. (16) yield, then

$$e \cos f \approx (3 - k) \frac{\Delta x}{x_0}, \quad e \sin f \approx \frac{1 - k}{x_0}. \tag{27}$$

Eqs. (27) imply, on the basis of $e > 0$ and $|e \sin f| \gg |e \cos f|$:

$$\begin{aligned}
\lim_{x \rightarrow x_0 - 2\pi} f &= \pi / 2, \quad k < 1, \\
\lim_{x \rightarrow x_0 - 2\pi} f &= 3\pi / 2, \quad k > 1.
\end{aligned} \tag{28}$$

5.1.5. $x = x_0 - 3\pi$

Putting $x = x_0 - 3\pi$ into the last of Eqs. (16) one obtains that the point $x = x_0 - 3\pi$ corresponds to: i) local maximum for the case $k < 3$, ii) local minimum for the case $k > 3$, for the function $e(t)$, or, $e(x)$. The value of the osculating eccentricity is

$$e^2(x = x_0 - 3\pi) \approx (5 - k)^2 x_0^{-2} + 3\pi (5k - 11) (5 - k) x_0^{-3}. \tag{29}$$

Comparison of Eqs. (22) and (29) leads to the conclusion

$$\begin{aligned}
e(x_0 - \pi) &= e(x_0 - 3\pi), \quad k = 11/5, 5 \\
e(x_0 - \pi) &< e(x_0 - 3\pi), \quad 11/5 < k < 5 \\
e(x_0 - \pi) &> e(x_0 - 3\pi), \quad k \in (11/5, 5)'.
\end{aligned} \tag{30}$$

As for time evolution of true anomaly, the results are analogous to Eqs. (23) – (24).

5.1.6. Discussion

The values of eccentricities are collected in Eqs. (26) and (30). The consequence of these equations is that mean values of eccentricity during the corresponding periods exhibit

similar properties. The cases $k = 11/5$, $k = 5$ yield the constant values of mean eccentricities during a long time evolution.

The case $k = 1$ was treated in detail in Klačka and Kaufmannová (1992). The main results are presented in Figs. 1, 3 and 8 in Klačka and Kaufmannová (1992).

We can collect the results for the cases $k = 1$ and $k = 5$ in the statement that true anomaly is a discontinuous function: i) at $x = x_0 - 2\pi m$, $m \in N$ for $k = 1$, i) at $x = x_0 - \pi(2m - 1)$, $m \in N$ for $k = 5$. We collect the results:

$$\begin{aligned} k = 1 : \quad \lim_{x \rightarrow x_0^-} f &= \pi, & \lim_{x \rightarrow (x_0 - \pi)} f &= 3\pi/2, \\ k = 1 : \quad \lim_{x \rightarrow (x_0 - 2\pi)^+} f &= 2\pi, & \lim_{x \rightarrow (x_0 - 2\pi)^-} f &= \pi, \end{aligned} \quad (31)$$

$$\begin{aligned} k = 5 : \quad \lim_{x \rightarrow x_0^-} f &= 3\pi/2, & \lim_{x \rightarrow (x_0 - \pi)^+} f &= 2\pi, \\ k = 5 : \quad \lim_{x \rightarrow (x_0 - \pi)^-} f &= \pi, & \lim_{x \rightarrow (x_0 - 2\pi)} f &= 3\pi/2. \end{aligned} \quad (32)$$

6. Terminal values of osculating elements

As for the terminal values of osculating elements presented in BJ-paper, the results may be collected in two important statements:

$$\lim_{x \rightarrow 0^+} f = \pi \quad ; \quad \lim_{x \rightarrow 0^+} e = 1. \quad (33)$$

Let us calculate other important quantities. The results are:

$$\begin{aligned} \lim_{x \rightarrow 0^+} r &= 0 \quad ; \quad \lim_{x \rightarrow 0^+} a = 0, \\ \lim_{x \rightarrow 0^+} v_T &= 0 \quad ; \quad \lim_{x \rightarrow 0^+} v_R = -\frac{c}{2} \frac{1 - \beta}{\beta}, \\ \lim_{x \rightarrow 0^+} H &= 0 \quad ; \quad \lim_{x \rightarrow 0^+} E = -\infty, \end{aligned} \quad (34)$$

where r is particle's distance from the central point mass, a – semimajor axis, v_T – transversal component of the velocity vector, $v_R \equiv \dot{r}$ – radial component of the velocity vector, H – angular momentum, E – total energy of the particle with respect to the central point mass.

The obtained results presented by Eqs. (33) and (34) yield important inconsistencies. The osculating trajectory is parabola ($e = 1$), the particle is situated at apocenter ($f = \pi$) and the total energy is $E = -\infty$. Normal result is that $E = 0$ for the case $e = 1$. So, there is something wrong with the results, it seems.

The results given by Eqs. (33) and (34) are correct as for mathematical point of view – mathematical solutions of the discussed limits, based on the mathematical solution presented in BJ-paper. However, we are not interested in mathematics as the main theme.

We are interested in physics. Thus, the important inconsistencies presented by Eqs. (33) and (34) should show that physics is not completely correct. Really, physics is incorrect.

It is a physical nonsense when a particle losses an unlimited energy within a finite time. The results presented by Eqs. (33) and (34) correspond to this nonphysical situation. A particle spirals toward $r = 0$ in a finite time and its potential energy decreases in an unlimited value.

Eqs. (34) yield a hint how to put the discussed inconsistencies into a correct physics.

Since

$$\lim_{x \rightarrow 0^+} v_R = -\frac{c}{2} \frac{1 - \beta}{\beta} < -c, \quad (35)$$

for $0 < \beta < 1/3$, we have to use complete form of the P-R effect – relativistic effect (Klačka 1992a, Eq. (140)).

Eq. (35) yields that the form of the equation of motion containing only first order of v/c could be acceptable only for $0 \leq 1 - \beta \ll 1$ – only in this case the requirement $v \ll c$ holds. However, the third Kepler’s law yields $T^2 = 4 \pi^2 a^3 \{\mu_{\beta=0} (1 - \beta)\}^{-1}$ and $\lim_{\beta \rightarrow 1^-} T = \infty$. Thus, the situation $0 \leq 1 - \beta \ll 1$ is not physically interesting. This situation is evident also from Eq. (30) in Klačka (1992c): $\lim_{\beta \rightarrow 1^-} e_{in} > 1$ – no inspiralling toward the center occurs.

The conclusion of this section states that the physics used in BJ-paper is not competent to say something about the terminal values of osculating elements. Any comparison of the statements for various initial conditions can be done only for $r_{final} \gg r_g \equiv 2GM/c^2$, $v \ll c$. (We refer also to Klačka 1994c.) As an example we may mention the time of inspiralling toward the point mass center, based on approximations in first order in v/c – the ‘time of inspiralling’ corresponds to $r_{final} \geq 200$ km for the mass of the Sun. As for the averaged equations for osculating elements, the condition (104) in Klačka 1992d must be fulfilled).

7. Conclusions

We have completed analytical formulae (expansions) for pseudo-circular orbits for the Poynting-Robertson effect.

The statement ‘The zero values of our arbitrary constants do not imply that the osculating $e = 0$, but they match the case of the zero mean eccentricity ...’ (Breiter and Jackson 1998, section 5 on page 240) is incorrect – it is not possible that any continuous non-negative quantity has zero mean value, unless it is identical zero.

Analytical results for more general pseudo-circular orbits than those discussed in (Breiter and Jackson 1998), were obtained in this paper.

We have shown that terminal values of osculating elements lead to serious inconsistencies caused by the fact that relativistic equations of motion only in first order in v/c were used in Breiter and Jackson (1998).

Finally, we have to stress several facts for elliptical orbits for the P-R effect containing only first order in v/c and in the zone of its applicability in two-body problem.

At first, the evident result is that osculating semi-major axis is still a decreasing function of time – energy decreases (see Eq. (22) in Klačka 1992c).

As for osculating eccentricity, it:

- i) may be an increasing function of time for an initial long-time interval if Eqs. (3) and (4) are fulfilled;
- ii) still alternates in an increasing and a decreasing functions on short-time intervals.

As for mean eccentricity, it may be:

- i) an increasing function of time (for a suitable initial conditions: see Fig. 3 in Klačka and Kaufmannová (1992), or, Eqs. (3) and (4) – $k \in (11/5, 5)$ for pseudocircular orbits);
- ii) a constant function of time (see Fig. 4 in Klačka and Kaufmannová (1992); moreover, the cases $k = 11/5$, $k = 5$ for pseudo-circular orbits);
- iii) a decreasing function of time for some initial time interval followed by an increasing function of time (Klačka and Kaufmannová (1992) – $k \in (11/5, 5)'$ for pseudo-circular orbits);
- iv) a decreasing function of time (Wyatt and Whipple 1950 – without derivation; correct derivation Klačka 1992c) for a long-time interval.

(One must be careful which type of osculating elements is used. The case ii) in mean eccentricity may corresponds to osculating elements defined by value μ_0 – osculating elements I in Klačka 1992c, or osculating elements in Klačka and Kaufmannová (1992). The cases i), iii), iv) and pseudo-circular orbits ii) in mean eccentricity correspond to osculating elements defined by value $\mu_0(1 - \beta)$ – osculating elements II in Klačka 1992c, or, ‘non-osculating’ elements in Klačka and Kaufmannová (1992).)

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